

Separation of variables for Gaudin-Calogero systems

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Abstract. *We construct an elliptic analogue of Sklyanin's separation of variables for the $sl(2)$ Gaudin system, using an adaptation of Drinfeld's Radon transformations.*

Introduction.

The geometric Langlands conjectures, as formulated by Beilinson and Drinfeld in [2], aim at the construction of certain \mathcal{D} -modules on the moduli space of G -bundles over (punctured) curves (G a reductive group). Positive characteristic versions of these conjectures had been solved earlier by Drinfeld in [3], in the case $G = GL_2$.

In the paper [6], the \mathcal{D} -modules arising from the construction of [2] were studied in the special case of a rational curve with marked points, and identified with the Gaudin model. Then, in [8], Drinfeld's construction of local systems on the moduli space of rank two vector bundles on a curve in positive characteristic ([3]) was adapted to the complex situation. The identification of these two constructions amounts to Sklyanin's separation of variables ([10]), as it was noticed in [8]. This computation is recalled in the first part of this text.

The question has been raised in [8] to construct a similar separation of variables for the Gaudin-Calogero systems, which were computed in [4] and [9], and play a similar role in the case of a punctured elliptic curve. This note aims at solving this question. In the present case Drinfeld's diagrams for Radon transformation have to be slightly modified.

It is also worth to note that the systems presented here, are the specialization at the critical level, of the Knizhnik-Zamolodchikov-Bernard equations on the torus. According to the general viewpoint that the diagrams of [3] are related to the Drinfeld-Sokolov reduction, the generalisation of the present work to noncritical level should relate these equations to the Virasoro correlators on the torus. In [5], the Bethe equations were connected with the unitarity property of the KZB equation on the torus. It would be desirable to clarify further the connection between these issues.

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1. Separation of variables for sl_2 Gaudin systems.

Let us begin with some reminders on the Gaudin system. Let $X = \mathbf{CP}^1$, z_α be marked points on X , $\alpha = 1, \dots, N$, G be $SL_2(\mathbf{C})$, $B \subset G$ be the upper triangular subgroup. The moduli space $\mathcal{M}_G(X, z_\alpha)$ of G -bundles on \mathbf{CP}^1 , with parabolic structures at z_α , is the disjoint union of the $\mathcal{M}_G^{(n)}(X, z_\alpha)$, $n \geq 0$, corresponding to the parabolic structures on the sheaf $\mathcal{O}(n\infty) \oplus \mathcal{O}(-n\infty)$. We then identify $\mathcal{M}_G^{(n)}(X, z_\alpha)$ with $P_n \backslash (G/B)^N$, $P_0 = G$, and $P_n = \left\{ \begin{pmatrix} t & p(z) \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbf{C}^\times, p(z) \in \mathbf{C}[z], \deg(p) \leq 2n \right\}$ for $n > 0$, P_0 acting diagonally and P_n acting on the α -th factor by left translation, after the replacement of z by z_α . In what follows we will deal with $\mathcal{M}_G^{(0)}(X, z_\alpha)$.

Let us fix weights, λ_α , $\alpha = 1, \dots, N$. On $\mathcal{M}_G^{(0)}(X, z_\alpha)$ lives the bundle $\mathcal{L}_{(\lambda_\alpha)}$, quotient of the bundle $\boxtimes_{\alpha=1}^N \mathcal{L}_{\lambda_\alpha}$ ($\mathcal{L}_{\lambda_\alpha}$ is the line bundle on G/B , corresponding to the weight λ_α). The natural action of $Z(U_{-2}\hat{sl}_2)_{loc}$ (the center of the local completion of the enveloping algebra of the central extension of $sl_2(\mathbf{C}((z)))$ at level -2) is by differential operators L_α , $\alpha = 1, \dots, N$, which were identified in [6] with the Gaudin hamiltonians $L_\alpha = \sum_{\beta \neq \alpha} \frac{I_a^{(\alpha)} I_a^{(\beta)}}{z_\alpha - z_\beta}$, I_a, I^a being an orthogonal basis of $sl_2(\mathbf{C})$.

Following the conjectures of [2], the \mathcal{D} -modules on $\mathcal{M}_G(X, z_\alpha)$ (twisted by $\mathcal{L}_{(\lambda_\alpha)}$) defined by $L_\alpha - \mu_\alpha$ should satisfy the Hecke eigenvalue property. In [3], a construction of such modules was given in the case of a curve of genus > 0 without punctures.

Following [8], let us show how Drinfeld's construction in [3] can be adapted for $X = \mathbf{CP}^1$ with marked points z_α . Consider the space

$$\mathcal{M}_B^{(0)}(X, z_\alpha) = \{(\text{parabolic structure on } \mathcal{O}_{\mathbf{CP}^1}^2 \text{ at } z_\alpha, \\ \text{class of morphisms } \mathcal{O}_{\mathbf{CP}^1} \rightarrow \mathcal{O}_{\mathbf{CP}^1}^2)\},$$

the morphisms being considered up to $\mathcal{O}_{\mathbf{CP}^1}$ -automorphisms; the cokernel of the morphism considered in this definition is $\mathcal{O}_{\mathbf{CP}^1}$. It is natural to consider $K_{\mathbf{CP}^1, z_\alpha} = K_{\mathbf{CP}^1}(\sum_{\alpha=1}^N (z_\alpha))$ as the canonical bundle in our punctured situation, and then the space $\{\text{classes of morphisms } K_{\mathbf{CP}^1, z_\alpha}^{-1} \rightarrow \mathcal{O}_{\mathbf{CP}^1}\}$, its mapping π to X^{N-2} (given by the zeroes of a given section) and the diagram

$$(1) \quad \begin{array}{ccc} & \mathcal{Z} & \\ & \swarrow p_0 & \searrow q_0 \\ \mathcal{M}_B^{(0)}(\mathbf{CP}^1, z_\alpha) & & \mathbf{P} \text{Hom}(K_{\mathbf{CP}^1, z_\alpha}^{-1}, \mathcal{O}_{\mathbf{CP}^1}) \xrightarrow{\pi} X^{(N-2)} \\ p \swarrow & \searrow & \swarrow \\ \mathcal{M}_G^{(0)}(\mathbf{CP}^1, z_\alpha) & & \{\mathcal{O}_{\mathbf{CP}^1}\} \end{array}$$

p being the projection on the first factor, and the correspondance \mathcal{Z} being defined to be the set of $((l_\alpha, i), j)$, l_α : line in the fiber of $\mathcal{O}_{\mathbf{CP}^1}^2$ at z_α , i : morphism $\mathcal{O}_{\mathbf{CP}^1} \rightarrow \mathcal{O}_{\mathbf{CP}^1}^2$, j : morphism $K_{\mathbf{CP}^1, z_\alpha} \rightarrow \mathcal{O}_{\mathbf{CP}^1}$, proportional i, j 's being considered

equivalent, such that denoting by $k : \mathcal{O}_{\mathbf{CP}^1}^2 \rightarrow \mathcal{O}_{\mathbf{CP}^1}$, the cokernel mapping of i , there exists a lift $j' : K_{\mathbf{CP}^1, z_\alpha} \rightarrow \mathcal{O}_{\mathbf{CP}^1}^2$ of j (i.e., we have $j = k \circ j'$), compatible with the parabolic structure (i.e., the image of j' at z_α should be the line l_α).

Let us fix weights λ_α for each α ; the \mathcal{D} -modules we will consider will be twisted by the following line bundles: $\mathcal{L}_{(\lambda_\alpha)}$ on $\mathcal{M}_G^{(0)}(X, z_\alpha)$, $p^*\mathcal{L}_{(\lambda_\alpha)}$ on $\mathcal{M}_B^{(0)}(X, z_\alpha)$, and $p_0^*p^*\mathcal{L}_{(\lambda_\alpha)}$ on \mathcal{Z} . For Y a variety and \mathcal{L} a line bundle on Y , we denote $(\mathcal{D}_Y)_\mathcal{L} = \mathcal{L} \otimes \mathcal{D}_Y \otimes \mathcal{L}^{-1}$. Let us fix now complex numbers $\mu_\alpha, \alpha = 1, \dots, N$, s.t. $\sum_{\alpha=1}^N \mu_\alpha = 0$, $\sum_{\alpha=1}^N \mu_\alpha z_\alpha + \sum_{\alpha=1}^N 2\lambda_\alpha(\lambda_\alpha - 1) = 0$, $\sum_{\alpha=1}^N \mu_\alpha z_\alpha^2 + \sum_{\alpha=1}^N 4\lambda_\alpha(\lambda_\alpha - 1)z_\alpha = 0$; we associate to them the operator on X ,

$$(2) \quad D_{(\lambda_\alpha), (\mu_\alpha)} = 2\partial_w^2 - \sum_{\alpha=1}^N \frac{\mu_\alpha}{w - z_\alpha} - \sum_{\alpha=1}^N \frac{2\lambda_\alpha(\lambda_\alpha - 1)}{(w - z_\alpha)^2},$$

and the \mathcal{D}_X -module $\mathcal{E}_{(\lambda_\alpha), (\mu_\alpha)} = \mathcal{D}_X / \mathcal{D}_X D_{(\lambda_\alpha), (\mu_\alpha)}$.

Consider on the other hand on $\mathcal{M}_G^{(0)}(X, z_\alpha)$ the twisted \mathcal{D} -module

$$\mathcal{M}_{(\mu_\alpha)} = \left(\mathcal{D}_{\mathcal{M}_G^{(0)}(X, z_\alpha)} \right)_{\mathcal{L}_{(\lambda_\alpha)}} / \sum_{\alpha=1}^N \left(\mathcal{D}_{\mathcal{M}_G^{(0)}(X, z_\alpha)} \right)_{\mathcal{L}_{(\lambda_\alpha)}} (L_\alpha - \mu_\alpha)$$

(the conditions on μ_α correspond to the relations on the L_α ,

$$\sum_{\alpha=1}^N L_\alpha = 0, \sum_{\alpha=1}^N L_\alpha z_\alpha + \sum_{\alpha=1}^N 2\lambda_\alpha(\lambda_\alpha - 1) = ef + fe + \frac{1}{2}h^2,$$

$$\sum_{\alpha=1}^N L_\alpha z_\alpha^2 + \sum_{\alpha=1}^N 4\lambda_\alpha(\lambda_\alpha - 1)z_\alpha = 2(e_1f + f_1e + \frac{1}{2}h_1h),$$

$e = \sum_{\alpha=1}^N e^{(\alpha)}$, $e_1 = \sum_{\alpha=1}^N z_\alpha e^{(\alpha)}$, analogous relations for f, f_1, h, h_1). We would like to show:

Proposition.— (cf. [8].) *There is a homomorphism of \mathcal{D} -modules*

$$\pi^* \mathcal{E}_{(\lambda_\alpha), (\mu_\alpha)}^{(N-2)} \rightarrow R(q_0)_* p_1^* p^* \mathcal{M}_{(\mu_\alpha)}[N],$$

which is an isomorphism over $\pi^{-1}((X - \{\infty\})^{(N-2)} - \Delta)$ (Δ is the diagonal part of $(X - \{\infty\})^{(N-2)}$).

(Here we denote, for \mathcal{F} a sheaf on a manifold V , by $\mathcal{F}^{(n)}$ the sheaf $(p_V)_*(\mathcal{F}^{\boxtimes n})$ on $V^{(n)} = V^n / S_n$, p_V being the projection $V^n \rightarrow V^{(n)}$.)

Proof. Let us give coordinates to the spaces of diagram (1).

$$\mathcal{M}_G^{(0)}(X, z_\alpha) \simeq G \backslash (G/B)^N = G \backslash (\mathbf{CP}^1)^N,$$

choosing the identification $\mathbf{CP}^1 \simeq G/B$, $t \mapsto \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix} B$, $0 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B$; then G acts on $(\mathbf{CP}^1)^N$ by homographic transformations. Now $\mathcal{M}_B^{(0)}(X, z_\alpha) = B \backslash (G/B)^N$; after fixing $\mathcal{O}_{\mathbf{CP}^1} \rightarrow \mathcal{O}_{\mathbf{CP}^1}^2$ to be $(1, 0)$, the lines l_α are $\mathbf{C}(1, t_\alpha^{-1})$, the t_α^{-1} being defined up to a global affine transformation. An element of $\text{Hom}(K_{\mathbf{CP}^1, z_\alpha}^{-1}, \mathcal{O}_{\mathbf{CP}^1})$ is a 1-form $\sum_{\alpha=1}^N \frac{u_\alpha dz}{z - z_\alpha}$, with $\sum_{\alpha=1}^N u_\alpha = 0$. The incidence relation defining \mathcal{Z} is $\sum_{\alpha=1}^N u_\alpha t_\alpha = 0$, since the first component of j' has to be $\sum_{\alpha=1}^N \frac{u_\alpha t_\alpha dz}{z - z_\alpha}$, and should be regular at ∞ . The map π associates to (u_α) , the solutions (w_i) of $\sum_{\alpha=1}^N \frac{u_\alpha}{z - z_\alpha} = 0$ (counting k times ∞ , if this function is $\sim c/w^{2+k}$ for $w \rightarrow \infty$, $c \neq 0$).

Let p_1 be the natural projection of $(G/B)^N$ on $G \backslash (G/B)^N$, then

$$p_1^*(\mathcal{M}_{(\mu_\alpha)}) = \mathcal{D}_{(G/B)^N} / \sum_{\alpha} \mathcal{D}_{(G/B)^N} (L_\alpha - \mu_\alpha) + \mathcal{D}_{(G/B)^N} sl_2(\mathbf{C}).$$

Introduce the formal variable z , then

$$\sum_{\alpha=1}^N \frac{L_\alpha - \mu_\alpha}{z - z_\alpha} + \sum_{\alpha=1}^N \frac{2\lambda_\alpha(\lambda_\alpha - 1)}{(z - z_\alpha)^2} = e(z)f(z) + f(z)e(z) + \frac{1}{2}h(z)^2 - \sum_{\alpha=1}^N \frac{\mu_\alpha}{z - z_\alpha}$$

with $e(z) = \sum_{\alpha=1}^N \frac{e^{(\alpha)}}{z - z_\alpha}$, etc., $e^{(\alpha)} = t_\alpha^2 \frac{\partial}{\partial t_\alpha} + 2\lambda_\alpha t_\alpha$, $f^{(\alpha)} = -\frac{\partial}{\partial t_\alpha}$, $h^{(\alpha)} = 2(t_\alpha \frac{\partial}{\partial t_\alpha} + \lambda_\alpha)$. The Radon transform of the \mathcal{D} -module generated by the $L_\alpha - \mu_\alpha$ is the \mathcal{D} -module generated by the $\bar{L}_\alpha - \mu_\alpha$, where

$$\sum_{\alpha=1}^N \frac{\bar{L}_\alpha - \mu_\alpha}{z - z_\alpha} + \sum_{\alpha=1}^N \frac{2\lambda_\alpha(\lambda_\alpha - 1)}{(z - z_\alpha)^2} = \bar{e}(z)\bar{f}(z) + \bar{f}(z)\bar{e}(z) + \frac{1}{2}\bar{h}(z)^2 - \sum_{\alpha=1}^N \frac{\mu_\alpha}{z - z_\alpha}$$

$\bar{e}(z) = \sum_{\alpha=1}^N \frac{\bar{e}^{(\alpha)}}{z - z_\alpha}$, analogous formulae for $\bar{f}(z)$, $\bar{h}(z)$, $\bar{e}^{(\alpha)} = -(u_\alpha(\frac{\partial}{\partial u_\alpha})^2 + 2(\lambda_\alpha + 1)\frac{\partial}{\partial u_\alpha})$, $\bar{f}^{(\alpha)} = u_\alpha$, $\bar{h}^{(\alpha)} = -2(u_\alpha \frac{\partial}{\partial u_\alpha} + \lambda_\alpha + 1)$.

Consider the operator

$$\hat{L}(w_i) = \sum_{\alpha=1}^N \frac{1}{w_i - z_\alpha} (\bar{L}_\alpha - \mu_\alpha),$$

and let $\hat{e}(w_i) = \sum_{\alpha=1}^N \frac{1}{w_i - z_\alpha} \bar{e}^{(\alpha)}$, analogous formulae for $\hat{f}(w_i)$, $\hat{h}(w_i)$. Then

$$\begin{aligned}
& \hat{L}(w_i) + \sum_{\alpha=1}^N \frac{\mu_\alpha}{w_i - z_\alpha} + \sum_{\alpha=1}^N \frac{2\lambda_\alpha(\lambda_\alpha - 1)}{(w_i - z_\alpha)^2} - [\hat{e}(w_i)\hat{f}(w_i) + \hat{f}(w_i)\hat{e}(w_i) + \frac{1}{2}\hat{h}(w_i)^2] \\
&= - \sum_{1 \leq \alpha, \beta \leq N} \frac{1}{w_i - z_\alpha} \{ [\bar{e}^{(\alpha)}, \frac{1}{w_i - z_\beta}] \bar{f}^{(\beta)} + [\bar{f}^{(\alpha)}, \frac{1}{w_i - z_\beta}] \bar{e}^{(\beta)} \\
&+ \frac{1}{2} [\bar{h}^{(\alpha)}, \frac{1}{w_i - z_\beta}] \bar{h}^{(\beta)} \} \\
&= - \sum_{1 \leq \alpha, \beta \leq N} - \frac{1}{w_i - z_\alpha} [u_\alpha \frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial u_\alpha} (\frac{1}{w_i - z_\beta})]_+ \cdot u_\beta \\
&+ \frac{2}{w_i - z_\alpha} u_\alpha \frac{\partial}{\partial u_\alpha} (\frac{1}{w_i - z_\beta}) (u_\beta \frac{\partial}{\partial u_\beta}) \\
&- \sum_{1 \leq \alpha, \beta \leq N} - \frac{1}{w_i - z_\alpha} \{ [2(\lambda_\alpha + 1) \frac{\partial}{\partial u_\alpha}, \frac{1}{w_i - z_\beta}] u_\beta \\
&- \frac{2}{w_i - z_\alpha} u_\alpha \frac{\partial}{\partial u_\alpha} (\frac{1}{w_i - z_\beta}) (\lambda_\beta + 1) \}
\end{aligned}$$

with $[a, b]_+ = ab + ba$. Now,

$$\begin{aligned}
& \sum_{\beta=1}^N \frac{\partial}{\partial u_\alpha} \left(\frac{1}{w_i - z_\beta} \right) u_\beta = - \frac{1}{w_i - z_\alpha}, \\
& \sum_{\beta=1}^N u_\alpha \left(\frac{\partial}{\partial u_\alpha} \right)^2 \left(\frac{1}{w_i - z_\beta} \right) u_\beta = -2u_\alpha \frac{\partial}{\partial u_\alpha} \left(\frac{1}{w_i - z_\alpha} \right),
\end{aligned}$$

so the last line gives zero, and the term in $[,]_+$ gives

$$\begin{aligned}
& -2 \sum_{\alpha=1}^N \frac{1}{w_i - z_\alpha} u_\alpha \frac{\partial}{\partial u_\alpha} \cdot \frac{1}{w_i - z_\alpha} + 2 \sum_{\alpha=1}^N \frac{1}{w_i - z_\alpha} u_\alpha \frac{\partial}{\partial u_\alpha} \left(\frac{1}{w_i - z_\alpha} \right) \\
&= -2 \sum_{\alpha=1}^N \frac{1}{(w_i - z_\alpha)^2} u_\alpha \frac{\partial}{\partial u_\alpha}.
\end{aligned}$$

(the dot denotes the product of differential operators). Then we deduce from

$$C \frac{\prod_{i=1}^{N-1} (z - w_i)}{\prod_{\alpha=1}^N (z - z_\alpha)} = \sum_{\alpha=1}^N \frac{u_\alpha}{z - z_\alpha},$$

$(\frac{dC}{C} + \sum_{i=1}^{N-2} \frac{dw_i}{w_i - z}) \sum_{\alpha=1}^N \frac{u_\alpha}{z - z_\alpha} = \sum_{\alpha=1}^N \frac{du_\alpha}{z - z_\alpha}$, so $du_\alpha = u_\alpha (\frac{dC}{C} + \sum_{i=1}^{N-2} \frac{dw_i}{w_i - z_\alpha})$ and

$$\frac{\partial}{\partial w_i} = \sum_{\alpha=1}^N \frac{u_\alpha}{w_i - z_\alpha} \frac{\partial}{\partial u_\alpha};$$

so the remaining term gives $-2 \sum_{\beta=1}^N \frac{\partial}{\partial w_i} \left(\frac{1}{w_i - z_\beta} \right) (u_\beta \frac{\partial}{\partial u_\beta}) = \sum_{\beta=1}^N \frac{2}{(w_i - z_\beta)^2} (u_\beta \frac{\partial}{\partial u_\beta})$.
Finally,

$$\hat{L}(w_i) = \hat{e}(w_i) \hat{f}(w_i) + \hat{f}(w_i) \hat{e}(w_i) + \frac{1}{2} \hat{h}(w_i)^2 - \sum_{\alpha=1}^N \frac{\mu_\alpha}{w_i - z_\alpha} - \sum_{\alpha=1}^N \frac{2\lambda_\alpha(\lambda_\alpha - 1)}{(w_i - z_\alpha)^2}.$$

Now, $\hat{f}(w_i) = 0$, and $\hat{h}(w_i) = -2[\frac{\partial}{\partial w_i} + A(w_i)]$, with $A(w_i) = \sum_{\alpha=1}^N \frac{\lambda_\alpha + 1}{w_i - z_\alpha}$, so

$$\hat{L}(w_i) = 2\left(\frac{\partial}{\partial w_i} + A(w_i)\right)^2 - \sum_{\alpha=1}^N \frac{\mu_\alpha}{w_i - z_\alpha} - \sum_{\alpha=1}^N \frac{2\lambda_\alpha(\lambda_\alpha - 1)}{(w_i - z_\alpha)^2}.$$

In this way, we have constructed an epimorphism from the $(N - 2)$ -th symmetric power of $\mathcal{E}_{(\lambda_\alpha), (\mu_\alpha)}$ (restricted to the complement of diagonals) to the \mathcal{D} -module generated by the $\bar{L}_\alpha - \mu_\alpha$'s (restricted to the complement of the discriminant), and so to the \mathcal{D} -module generated by the $\bar{L}_\alpha - \mu_\alpha$'s and the action of $sl_2(\mathbf{C})$.

Let us now show that it induces an isomorphism of the sheaves of local analytic solutions. Let us start with a local homomorphism of the first sheaf to the (analytic) structure sheaf. It is some function $(\psi(w_i))_{1 \leq i \leq N-2}$, such that $(\partial_{w_j}^2 + q(w_j))\psi(w_i) = 0$, for all j . We deduce from that relation, using our previous computations,

$$(3) \quad \sum_{i=1}^N \frac{(\bar{L}_\alpha - \mu_\alpha)\psi}{w_i - z_\alpha} = 0, \quad i = 1, \dots, N - 2.$$

We will consider ψ as a distribution on the space of all $(u_i)_{1 \leq i \leq N}$, supported on the hyperplane $\sum_{i=1}^N u_i = 0$, and analytic on this hyperplane. We obtain from (3)

$$(4) \quad (\bar{L}_\alpha - \mu_\alpha)\psi = u_\alpha \phi + z_\alpha u_\alpha \rho,$$

ϕ and ρ being distributions of the same nature as ψ ; indeed, ϕ and ρ can be obtained solving a Cramer system (since everywhere on the support of ψ , we can find two indices $\alpha \neq \beta$ such that $u_\alpha u_\beta \neq 0$), and \bar{L}_α 's commute with $\sum_{\alpha=1}^N u_\alpha$. From the relation $\sum_{\alpha=1}^N \bar{L}_\alpha = 0$ follows that $\rho = 0$; from $\sum_{\alpha=1}^N \bar{L}_\alpha z_\alpha + \sum_{\alpha=1}^N 2\lambda_\alpha(\lambda_\alpha - 1) = \bar{e}\bar{f} + \bar{f}\bar{e} + \frac{1}{2}\bar{h}^2$ and $\sum_{\alpha=1}^N \bar{L}_\alpha z_\alpha^2 + \sum_{\alpha=1}^N 4\lambda_\alpha(\lambda_\alpha - 1)z_\alpha = 2(\bar{e}_1\bar{f} + \bar{f}_1\bar{e} + \frac{1}{2}\bar{h}_1\bar{h})$, follows that $(\sum_{\alpha=1}^N u_\alpha)\bar{e}\psi = (\sum_{\alpha=1}^N z_\alpha u_\alpha)\phi$, and $(\sum_{\alpha=1}^N z_\alpha u_\alpha)\bar{e}\psi = (\sum_{\alpha=1}^N z_\alpha^2 u_\alpha)\phi$. So, on the complement of $\{(u_\alpha) | \sum_{\alpha=1}^N z_\alpha u_\alpha = 0 \text{ or } \sum_{\alpha=1}^N z_\alpha^2 u_\alpha = 0 \text{ or } (\sum_{\alpha=1}^N z_\alpha u_\alpha)^2 - (\sum_{\alpha=1}^N u_\alpha)(\sum_{\alpha=1}^N z_\alpha^2 u_\alpha) = 0\}$, ϕ and $\bar{e}\psi$ will vanish. So these distributions would have to be supported on a subvariety of the set of all (u_α) of codimension ≥ 2 , which is impossible. So we will have

$$(5) \quad (\bar{L}_\alpha - \mu_\alpha)\psi = 0, \quad \bar{e}\psi = 0.$$

Since $\bar{f}\psi = \bar{h}\psi = 0$ by construction, we have shown that ψ can be considered as a local homomorphism of the \mathcal{D} -module generated by the $\bar{L}_\alpha - \mu_\alpha$ (restricted to the complement of the discriminant) to the structure sheaf.

These two morphisms are clearly inverse to each other; if we show that both \mathcal{D} -modules have their characteristic varieties supported on the zero section, this will prove the proposition.

For $\mathcal{E}_{(\lambda_\alpha), (\mu_\alpha)}^{(N-2)}$ it is clear, since it is true for $\mathcal{E}_{(\lambda_\alpha), (\mu_\alpha)}$. On the other hand, the characteristic variety of $R(q_0)_* p_1^* p^* \mathcal{M}_{(\mu_\alpha)}[N]$ is the set of (u_i, ξ_i) , with $\sum_{i=1}^N u_i = 0$ and up to equivalence $(u_i, \xi_i) \sim (u_i, \xi_i + \lambda)$, and

$$\sum_{i=1}^N \frac{u_i}{z - z_i} = \left(\sum_{i=1}^N \frac{u_i \xi_i}{z - z_i} \right) \left(\sum_{i=1}^N \frac{u_i \xi_i^2}{z - z_i} \right).$$

This equation gives

$$\sum_{i=1}^N \frac{u_i}{z - z_i} = \frac{RA^2(z)}{\prod_{i=1}^N (z - z_i)}, \sum_{i=1}^N \frac{u_i \xi_i}{z - z_i} = \frac{RAB(z)}{\prod_{i=1}^N (z - z_i)}, \sum_{i=1}^N \frac{u_i \xi_i^2}{z - z_i} = \frac{RB^2(z)}{\prod_{i=1}^N (z - z_i)},$$

R, A, B polynomials. The set $\{w_i, 1 \leq i \leq N-2\}$ is the union of the set of zeroes of RA^2 , and of ∞ counted $N-2 - \deg RA^2$ times. Since the w_i 's are pairwise distinct, we have $A = \text{const.}$, and $\deg R = N-2$ or $N-3$. Since no w_i coincides with ∞ , $\deg R = N-2$; so, B is also constant, and the ξ_i 's are all equal; but this is equivalent to $\xi_i = 0$. ■

2. Separation of variables for the sl_2 Gaudin-Calogero system

Let X be the elliptic curve $\mathbf{C}^\times/q^{\mathbf{Z}}$, with marked points $z_\alpha q^{\mathbf{Z}}$, $\alpha = 1, \dots, N$. In the sl_2 case, the Gaudin-Calogero system (which plays the role of the Gaudin system in the present situation, cf. [4], [9]) takes place in the space

$$\begin{aligned} \mathcal{M}_G^{(0)}(X, z_\alpha) &= \{(\mathcal{E}_{(t, t^{-1})}, \text{parabolic structure at } z_\alpha \text{ given by } t_\alpha \in \mathbf{CP}^1)\} / \\ &\quad [(t, t_\alpha) \sim (qt, z_\alpha t_\alpha), (t, t_\alpha) \sim (t, ut_\alpha), u \in \mathbf{C}^\times], t \in \mathbf{C}^\times] \\ &= \mathbf{C}^\times \times (\mathbf{CP}^1)^N / \mathbf{C}^\times \ltimes \mathbf{Z}^N. \end{aligned}$$

Here $\mathcal{E}_{(t_1, \dots, t_n)}$ is the bundle on X defined by $\mathbf{C}^\times \times \mathbf{C}^n / [(z, \xi) \sim (qz, \text{diag}(t_i)\xi)]$, for $t_1, \dots, t_n \in \mathbf{C}^\times$.

We consider then the space

$$\begin{aligned} \mathcal{M}_B^{(0)}(X, z_\alpha) &= \{(\mathcal{E}_{(t, t^{-1})}, j : \mathcal{E}_t \rightarrow \mathcal{E}_{(t, t^{-1})}, \text{par. str. given by } t_\alpha \in \mathbf{CP}^1), t \in \mathbf{C}^\times\} / \\ &\quad [j \sim \lambda j, \lambda \in \mathbf{C}^\times, (t, t_\alpha) \sim (qt, z_\alpha t_\alpha), (t, t_\alpha) \sim (t, ut_\alpha), u \in \mathbf{C}^\times]; \end{aligned}$$

it has a natural projection p to $\mathcal{M}_G^{(0)}(X, z_\alpha)$. Consider now the diagram

(7)

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & p_1 \swarrow & & \searrow q_1 & \\ \mathcal{M}_B^{(0)}(X, z_\alpha) & & & & \{\mathcal{E} \in \text{Pic}^0(X), \omega \in \mathbf{P} \text{Hom}(K_{X, z_\alpha}^{-1} \otimes \mathcal{E}^{-1}, \mathcal{E})\} \\ & p \swarrow & p_0 \searrow & \swarrow q_0 & \searrow \pi \\ \mathcal{M}_G^{(0)}(X, z_\alpha) & & \text{Pic}^0(X) & & X^{(N)} \end{array}$$

where $K_{X,z_\alpha} = \Omega_X^1(\sum_{\alpha=1}^N(z_\alpha))$, p_0 is the projection

$$\text{class}(\mathcal{E}_{(t,t^{-1})}, j : \mathcal{E}_t \rightarrow \mathcal{E}_{(t,t^{-1})}, \text{ par. str.}) \mapsto \mathcal{E}_{t^{-1}}$$

(p_0 associates to j its cokernel), q_0 associates \mathcal{E} to (\mathcal{E}, ω) , π associates to (\mathcal{E}, ω) the set of zeroes of ω , and \mathcal{Z} is the incidence variety, defined by the conditions that ω lifts to a morphism $j' : K_{X,z_\alpha}^{-1} \otimes \mathcal{E}_{t^{-1}} \rightarrow \mathcal{E}_{(t,t^{-1})}$, compatible with the parabolic structure. Writing

$$\omega = \sum_{\alpha=1}^N u_\alpha \frac{\theta(t^{-2} z z_\alpha^{-1})}{\theta(t^{-2}) \theta(z z_\alpha^{-1})} \frac{dz}{z},$$

the first component of j' has to be $\sum_{\alpha=1}^N u_\alpha t_\alpha \frac{\dot{\theta}(z z_\alpha^{-1})}{\theta(z z_\alpha^{-1})} \frac{dz}{z}$, so that the incidence condition is $\sum_{\alpha=1}^N u_\alpha t_\alpha = 0$.

(Fix our conventions for θ - and \wp -functions: $\theta(z) = \prod_{i \geq 0} (1 - q^i z) \prod_{i > 0} (1 - q^i z^{-1})$, $\wp(\ln z) = -(\frac{\dot{\theta}}{\theta})(z)$, so $\wp(\tau) \sim \tau^{-2} + \dots$ for $\tau \rightarrow 0$; we denote $\dot{f}(z) = z \frac{df}{dz}$.)

The lift to $\mathbf{C}^\times \times (\mathbf{CP}^1)^N$ of the Gaudin-Calogero operators ([4], [9]) are defined as follows: let z be a formal variable, belonging to $X - \{z_\alpha\}$. We have

$$\begin{aligned} L(z) &= e(z)f(z) + f(z)e(z) + \frac{1}{2}h(z)^2 \\ &= L_0 + \sum_{\alpha=1}^N L_\alpha \frac{\dot{\theta}}{\theta}(z z_\alpha^{-1}) + \sum_{\alpha=1}^N 2\lambda_\alpha(\lambda_\alpha - 1)\wp(\ln z z_\alpha^{-1}) \\ &\quad + \sum_{\alpha=1}^N \frac{1}{2}h^{(\alpha)}\left(\sum_{\alpha=1}^N h^{(\alpha)}\right)\left(\frac{\dot{\theta}}{\theta}(z z_\alpha^{-1})\right)^2, \end{aligned}$$

where $e(z) = \sum_{\alpha=1}^N \frac{\theta(t^{-2} z z_\alpha^{-1})}{\theta(t^{-2}) \theta(z z_\alpha^{-1})} e^{(\alpha)}$, $h(z) = 2t^2 \frac{\partial}{\partial t^2} + 2k \frac{\dot{\theta}}{\theta}(t^2) + \sum_{\alpha=1}^N \frac{\dot{\theta}}{\theta}(z z_\alpha^{-1}) h^{(\alpha)}$, $f(z) = \sum_{\alpha=1}^N \frac{\theta(t^2 z z_\alpha^{-1})}{\theta(t^2) \theta(z z_\alpha^{-1})} f^{(\alpha)}$, and $e^{(\alpha)} = t_\alpha^2 \frac{\partial}{\partial t_\alpha} + 2\lambda_\alpha t_\alpha$, $f^{(\alpha)} = -\frac{\partial}{\partial t_\alpha}$, $h^{(\alpha)} = 2(t_\alpha \frac{\partial}{\partial t_\alpha} + \lambda_\alpha)$.

Let us fix now complex numbers μ_α , $\alpha = 0, \dots, N$, with $\sum_{\alpha=1}^N \mu_\alpha = 0$ (this condition corresponds to the fact that $\sum_{\alpha=1}^N L_\alpha$ belongs to the left ideal generated by $\sum_{\alpha=1}^N h^{(\alpha)}$); consider on $\mathcal{M}_G^{(0)}(X, z_\alpha)$, the \mathcal{D} -module (twisted by the quotient $\mathcal{L}_{k,(\lambda_\alpha)}$ of $i^{-1}(\mathcal{L}_k^{\boxtimes 2}) \boxtimes \boxtimes_{\alpha=1}^N \mathcal{L}_{\lambda_\alpha}$, $i : \text{Ker}(s) \rightarrow X^{(2)}$, $s : X^{(2)} \rightarrow X$ the sum mapping, \mathcal{L}_k a bundle of degree k on X),

$$\mathcal{M}_{(\mu_\alpha)} = \left(\mathcal{D}_{\mathcal{M}_G^{(0)}(X, z_\alpha)} \right)_{\mathcal{L}_{k,(\lambda_\alpha)}} / \sum_{\alpha=0}^N \left(\mathcal{D}_{\mathcal{M}_G^{(0)}(X, z_\alpha)} \right)_{\mathcal{L}_{k,(\lambda_\alpha)}} (L_\alpha - \mu_\alpha).$$

Consider then the operator on $X - \{z_\alpha\}$,

$$D_{(\lambda_\alpha),(\mu_\alpha)} = 2 \left(w \frac{\partial}{\partial w} \right)^2 - \mu_0 - \sum_{\alpha=1}^N \mu_\alpha \frac{\dot{\theta}}{\theta}(w z_\alpha^{-1}) + 2 \sum_{\alpha=1}^N \lambda_\alpha(\lambda_\alpha - 1) \wp(\ln w z_\alpha^{-1}),$$

and the \mathcal{D}_X -module

$$\mathcal{E}_{(\lambda_\alpha),(\mu_\alpha)} = \mathcal{D}_X / \mathcal{D}_X D_{(\lambda_\alpha),(\mu_\alpha)}.$$

The fibration π has fibers \mathbf{C}^\times ; we will twist inverse images under π by the function $C^{\sum_{\alpha=1}^N (\lambda_\alpha + 1)}$ (C coordinate on the fiber). Also, we will work with $k = 0$ (multiplication by $\theta(t^2)^k$ taking us back to this case). We will show that:

Proposition.— *There is a homomorphism of \mathcal{D} -modules from the twisted inverse image $\pi^* \mathcal{E}_{(\lambda_\alpha),(\mu_\alpha)}^{(N)}$ to $R(q_1)_* p_1^* p^* \mathcal{M}_{(\mu_\alpha)}[N]$, which is an isomorphism over $\pi^{-1}(X^{(N)} - \Delta)$ (Δ is the diagonal part of $X^{(N)}$).*

Proof. Let

$$p_1 : \mathbf{C}^\times \times (\mathbf{CP}^1)^N \rightarrow \mathbf{C}^\times \times (\mathbf{CP}^1)^N / \mathbf{C}^\times \ltimes \mathbf{Z}^N$$

be the natural projection, then

$$p_1^*(\mathcal{M}_{(\mu_\alpha)}) = \mathcal{D}_{\mathbf{C}^\times \times (\mathbf{CP}^1)^N} / \sum_{\alpha} \mathcal{D}_{\mathbf{C}^\times \times (\mathbf{CP}^1)^N} (L_\alpha - \mu_\alpha) + \mathcal{D}_{\mathbf{C}^\times \times (\mathbf{CP}^1)^N} \left(\sum_{\alpha} h^{(\alpha)} \right).$$

Because of the factor $\mathcal{D}_{\mathbf{C}^\times \times (\mathbf{CP}^1)^N} (\sum_{\alpha} h^{(\alpha)})$, the \mathcal{D} -module is constant along the fibers of the action of \mathbf{C}^\times . Its Radon transform is the \mathcal{D} -module generated by $\bar{L}_\alpha - \mu_\alpha$ and $\sum_{\alpha=1}^N \bar{h}^{(\alpha)}$, where

$$\begin{aligned} \bar{L}_0 + \sum_{\alpha=1}^N \bar{L}_\alpha \frac{\dot{\theta}}{\theta} (zz_\alpha^{-1}) + \sum_{\alpha=1}^N 2\lambda_\alpha (\lambda_\alpha - 1) \wp(\ln zz_\alpha^{-1}) + \sum_{\alpha=1}^N \frac{1}{2} \bar{h}^{(\alpha)} \left(\sum_{\alpha=1}^N \bar{h}^{(\alpha)} \right) \left(\frac{\dot{\theta}}{\theta} (zz_\alpha^{-1}) \right)^2 \\ = \bar{e}(z) \bar{f}(z) + \bar{f}(z) \bar{e}(z) + \frac{1}{2} \bar{h}(z)^2, \end{aligned}$$

$\bar{e}(z) = \sum_{\alpha=1}^N \frac{\theta(t^{-2}zz_\alpha^{-1})}{\theta(t^{-2})\theta(zz_\alpha^{-1})} \bar{e}^{(\alpha)}$, analogous formulae for $\bar{f}(z)$ and $\bar{h}(z)$, with

$$\bar{e}^{(\alpha)} = -[u_\alpha \left(\frac{\partial}{\partial u_\alpha} \right)^2 + 2(\lambda_\alpha + 1) \frac{\partial}{\partial u_\alpha}], \bar{f}^{(\alpha)} = u_\alpha, \bar{h}^{(\alpha)} = -2[u_\alpha \frac{\partial}{\partial u_\alpha} + (\lambda_\alpha + 1)].$$

Consider the operator

$$\hat{L}(w_i) = \bar{L}_0 - \mu_0 + \sum_{\alpha=1}^N \frac{\dot{\theta}}{\theta} (w_i z_\alpha^{-1}) (\bar{L}_\alpha - \mu_\alpha) + \sum_{\alpha=1}^N \left(\frac{\dot{\theta}}{\theta} (w_i z_\alpha^{-1}) \right)^2 \frac{1}{2} \bar{h}^{(\alpha)} \left(\sum_{\alpha=1}^N \bar{h}^{(\alpha)} \right).$$

Let $\hat{e}(w_i) = \sum_{\alpha=1}^N \frac{\theta(t^{-2}w_i z_\alpha^{-1})}{\theta(t^{-2})\theta(w_i z_\alpha^{-1})} \bar{e}^{(\alpha)}$, etc. Let us compute the difference

$$\begin{aligned} -\hat{L}(w_i) + (\hat{e}(w_i) \hat{f}(w_i) + \hat{f}(w_i) \hat{e}(w_i) + \frac{1}{2} \hat{h}(w_i)^2 - \mu_0 - \sum_{\alpha=1}^N \mu_\alpha \frac{\dot{\theta}}{\theta} (w_i z_\alpha^{-1}) \\ + 2\lambda_\alpha (\lambda_\alpha - 1) \wp(\ln w_i z_\alpha^{-1})); \end{aligned}$$

it is equal to

$$(8) \quad \sum_{\alpha, \beta=1}^N \frac{\theta(t^{-2}w_i z_\alpha^{-1})}{\theta(t^{-2})\theta(w_i z_\alpha^{-1})} [\bar{e}^{(\alpha)}, \frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})}] \bar{f}^{(\beta)} \\ + \frac{1}{2} \sum_{\alpha, \beta=1}^N \frac{\dot{\theta}}{\theta}(w_i z_\alpha^{-1}) [\bar{h}^{(\alpha)}, \frac{\dot{\theta}}{\theta}(w_i z_\beta^{-1})] \bar{h}^{(\beta)}.$$

The first term of (8) is the sum of (9) and (10), where

$$(9) = - \sum_{\alpha, \beta=1}^N \frac{\theta(t^{-2}w_i z_\alpha^{-1})}{\theta(t^{-2})\theta(w_i z_\alpha^{-1})} [2u_\alpha \frac{\partial}{\partial u_\alpha} (\frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})}) \frac{\partial}{\partial u_\alpha} \\ + u_\alpha (\frac{\partial}{\partial u_\alpha})^2 (\frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})})] \cdot u_\beta \\ = - \sum_{\alpha=1}^N \frac{\theta(t^{-2}w_i z_\alpha^{-1})}{\theta(t^{-2})\theta(w_i z_\alpha^{-1})} 2u_\alpha [\sum_{\beta=1}^N \frac{\partial}{\partial u_\alpha} (\frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})}) u_\beta] \frac{\partial}{\partial u_\alpha} \\ - \sum_{\alpha=1}^N \frac{\theta(t^{-2}w_i z_\alpha^{-1})}{\theta(t^{-2})\theta(w_i z_\alpha^{-1})} 2u_\alpha \frac{\partial}{\partial u_\alpha} (\frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})}) \frac{\partial}{\partial u_\alpha} \\ - \sum_{\alpha=1}^N \frac{\theta(t^{-2}w_i z_\alpha^{-1})}{\theta(t^{-2})\theta(w_i z_\alpha^{-1})} u_\alpha \sum_{\beta=1}^N (\frac{\partial}{\partial u_\alpha})^2 (\frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})}) u_\beta.$$

and

$$(10) = \sum_{\alpha, \beta=1}^N \frac{\theta(t^{-2}w_i z_\alpha^{-1})}{\theta(t^{-2})\theta(w_i z_\alpha^{-1})} (-2)(\lambda_\alpha + 1) \frac{\partial}{\partial u_\alpha} (\frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})}) u_\beta \\ = \sum_{\alpha=1}^N 2(\lambda_\alpha + 1) \frac{\theta(t^{-2}w_i z_\alpha^{-1})}{\theta(t^{-2})\theta(w_i z_\alpha^{-1})} \frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})} \\ = \sum_{\alpha=1}^N 2(\lambda_\alpha + 1) [\wp(\ln t^2) - \wp(\ln w_i z_\alpha^{-1})]$$

We have

$$\sum_{\beta=1}^N \frac{\partial}{\partial u_\alpha} (\frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})}) u_\beta = - \frac{\theta(t^2 w_i z_\alpha^{-1})}{\theta(t^2)\theta(w_i z_\alpha^{-1})},$$

and

$$\sum_{\beta=1}^N (\frac{\partial}{\partial u_\alpha})^2 (\frac{\theta(t^2 w_i z_\beta^{-1})}{\theta(t^2)\theta(w_i z_\beta^{-1})}) u_\beta = -2 \frac{\partial}{\partial u_\alpha} (\frac{\theta(t^2 w_i z_\alpha^{-1})}{\theta(t^2)\theta(w_i z_\alpha^{-1})}).$$

So

$$\begin{aligned}
(9) &= \sum_{\alpha=1}^N 2u_{\alpha} \frac{\theta(t^2 w_i z_{\alpha}^{-1})}{\theta(t^2) \theta(w_i z_{\alpha}^{-1})} \frac{\theta(t^{-2} w_i z_{\alpha}^{-1})}{\theta(t^{-2}) \theta(w_i z_{\alpha}^{-1})} \frac{\partial}{\partial u_{\alpha}} \\
&= \sum_{\alpha=1}^N (-2) [\wp(\ln w_i z_{\alpha}^{-1}) - \wp(\ln t^2)] u_{\alpha} \frac{\partial}{\partial u_{\alpha}}.
\end{aligned}$$

The second term of (8) is the sum of (11) and (12), with

$$(11) = 2 \sum_{\alpha, \beta=1}^N \frac{\dot{\theta}}{\theta}(w_i z_{\alpha}^{-1}) u_{\alpha} \frac{\partial}{\partial u_{\alpha}} \left(\frac{\dot{\theta}}{\theta}(w_i z_{\beta}^{-1}) \right) u_{\beta} \frac{\partial}{\partial u_{\beta}}$$

and

$$\begin{aligned}
(12) &= \frac{1}{2} \sum_{\alpha, \beta=1}^N \frac{\dot{\theta}}{\theta}(w_i z_{\alpha}^{-1}) 2u_{\alpha} \frac{\partial}{\partial u_{\alpha}} \left(\frac{\dot{\theta}}{\theta}(w_i z_{\beta}^{-1}) \right) 2(\lambda_{\beta} + 1) \\
&= \sum_{\alpha=1}^N (\lambda_{\alpha} + 1) \left\{ -2w_i \frac{\partial}{\partial w_i} \left[\frac{\dot{\theta}}{\theta}(w_i z_{\alpha}^{-1}) \right] \right\} \\
&= \sum_{\alpha=1}^N 2(\lambda_{\alpha} + 1) \wp(\ln(w_i z_{\alpha}^{-1}))
\end{aligned}$$

To compute (11), we express the relation between the $\frac{\partial}{\partial w_i}$ and the $\frac{\partial}{\partial u_{\alpha}}$: we have

$$\sum_{\alpha=1}^N u_{\alpha} \frac{\theta(t^2 z z_{\alpha}^{-1})}{\theta(t^2) \theta(z z_{\alpha}^{-1})} = C \frac{\prod_{i=1}^N \theta(z w_i^{-1})}{\prod_{\alpha=1}^N \theta(z z_{\alpha}^{-1})},$$

so

$$\begin{aligned}
\sum_{\alpha=1}^N du_{\alpha} \frac{\theta(t^2 z z_{\alpha}^{-1})}{\theta(t^2) \theta(z z_{\alpha}^{-1})} + \frac{dt}{t} u_{\alpha} \left[\frac{\dot{\theta}}{\theta}(t^2 z z_{\alpha}^{-1}) - \frac{\dot{\theta}}{\theta}(t^2) \right] \frac{\theta(t^2 z z_{\alpha}^{-1})}{\theta(t^2) \theta(z z_{\alpha}^{-1})} \\
= C \frac{\prod_{i=1}^N \theta(z w_i^{-1})}{\prod_{\alpha=1}^N \theta(z z_{\alpha}^{-1})} \left[\frac{dC}{C} - \sum_{i=1}^N \frac{dw_i}{w_i} \frac{\dot{\theta}}{\theta}(z w_i^{-1}) \right];
\end{aligned}$$

by inspection of the pole at z_{α} , and because of $C \frac{\prod_{i=1}^N \theta(z_{\alpha} w_i^{-1})}{\prod_{\beta \neq \alpha} \theta(z_{\alpha} z_{\beta}^{-1})} = u_{\alpha}$, it follows that

$$du_{\alpha} = u_{\alpha} \left[\frac{dC}{C} - \sum_{i=1}^N \frac{dw_i}{w_i} \frac{\dot{\theta}}{\theta}(z_{\alpha} w_i^{-1}) \right],$$

and so

$$w_i \frac{\partial}{\partial w_i} = - \sum_{\alpha=1}^N \frac{\dot{\theta}}{\theta}(z_{\alpha} w_i^{-1}) u_{\alpha} \frac{\partial}{\partial u_{\alpha}}.$$

Now

$$(11) = 2 \sum_{\beta=1}^N -w_i \frac{\partial}{\partial w_i} \left(\frac{\dot{\theta}}{\theta} (z_\beta w_i^{-1}) \right) u_\beta \frac{\partial}{\partial u_\beta} = 2 \sum_{\beta=1}^N \wp(\ln w_i z_\beta^{-1}) u_\beta \frac{\partial}{\partial u_\beta}.$$

Finally

$$(9) + (11) = 2\wp(\ln t^2) \sum_{\alpha=1}^N u_\alpha \frac{\partial}{\partial u_\alpha} = 2\wp(\ln t^2) C \frac{\partial}{\partial C},$$

and

$$(10) + (12) = \sum_{\alpha=1}^N 2(\lambda_\alpha + 1) \wp(\ln t^2).$$

The sum of these terms is $2\wp(\ln t^2)(C \frac{\partial}{\partial C} + \sum_{\alpha=1}^N (\lambda_\alpha + 1))$; the term $C \frac{\partial}{\partial C} + \sum_{\alpha=1}^N (\lambda_\alpha + 1)$ (equal to $-\frac{1}{2} \sum_{\alpha=1}^N \bar{h}^{(\alpha)}$) is set to zero in the twisted inverse image $\pi^* \mathcal{E}_{(\lambda_\alpha), (\mu_\alpha)}^{(N)}$. It follows that

$$\begin{aligned} \hat{L}(w_i) &= 2[t^2 \frac{\partial}{\partial t^2} + w_i \frac{\partial}{\partial w_i} + A(w_i)]^2 - \mu_0 - \sum_{\alpha=1}^N \mu_\alpha \frac{\dot{\theta}}{\theta} (w_i z_\alpha^{-1}) \\ &\quad + \sum_{\alpha=1}^N 2\lambda_\alpha (\lambda_\alpha - 1) \wp(\ln w_i z_\alpha^{-1}), \end{aligned}$$

since $\hat{e}(w_i) = 0$, and $\hat{h}(w_i) = 2[t^2 \frac{\partial}{\partial t^2} + k \frac{\dot{\theta}}{\theta} (t^2) + w_i \frac{\partial}{\partial w_i} + A(w_i)]$, with $A(w_i) = -\sum_{\alpha=1}^N (\lambda_\alpha + 1) \frac{\dot{\theta}}{\theta} (w_i z_\alpha^{-1})$. The addition of the term $A(w_i)$ corresponds to the twisting by a $GL(1)$ -connection and does not change the $PGL(2)$ -oper, as in [8]. The addition of the term $t^2 \partial / \partial (t^2)$ corresponds to the fact that we are working here with the variables (w_i, t) , which are linked by the relation $t^2 \prod_{i=1}^N w_i = \prod_{\alpha=1}^N z_\alpha \pmod{q^{\mathbf{Z}}}$. Our statement follows as before. ■

Remark. As in the rational case, one may remark that the conditions on the μ_α 's to satisfy the Bethe ansatz equations, that can be found in [7], can be translated into the condition on the projective connection defined by them, to have a single-valued solution $\psi(z) = \prod \theta(za_i) / \prod_{\alpha=1}^n \theta(zz_\alpha^{-1})^{\lambda_\alpha}$. Thanks to the Leray formulae for Radon transformation [1], one could expect the Bethe eigenvectors to be expressed in the form

$$\begin{aligned} \Psi(t_1, \dots, t_N) = \int_{\Gamma} \frac{C^{-\sum_{\alpha=1}^N (\lambda_\alpha + 1)} \psi(w_1) \cdots \psi(w_N)}{(\sum_{\alpha=1}^N u_\alpha t_\alpha)^k} \\ \cdot \sum_{\alpha=1}^N (-1)^\alpha u_\alpha du_1 \wedge \cdots \wedge d\check{u}_\alpha \wedge \cdots \wedge du_N, \end{aligned}$$

$1 \leq k - \sum_{\alpha=1}^N (\lambda_\alpha + 1) \leq N - 1$, the integration being on a suitable cycle in \mathbf{CP}^N . In the general case, this formula should lead to the computation of the monodromy of the Gaudin-Calogero system (by deformation of the cycle of integration). It might be interesting to express this monodromy representation directly in terms of the one of the projective connection associated to the μ_α 's.

References

- [1] A. d'Agnolo, P. Schapira, *Quantification de Leray de la dualité projective*, C. R. Acad. Sci. Paris, t.139, série I (1994), 595-8.
- [2] A.A. Beilinson, V.G. Drinfeld, *Quantization of Hitchin's fibration and Langlands program*, preprint.
- [3] V.G. Drinfeld, *Two-dimensional representations of the fundamental group of a curve over a finite field and automorphic forms on $GL(2)$* , Amer. J. Math. 105 (1983), 85-114.
- [4] B. Enriquez, V. Rubtsov, *Hitchin systems, higher Gaudin operators and r -matrices*, preprint alg-geom/9503010.
- [5] F. Falceto, K. Gawedski, *Unitarity of the Knizhnik-Zamolodchikov-Bernard connection and the Bethe Ansatz for the elliptic Hitchin system*, hep-th/9604094.
- [6] B.L. Feigin, E.V. Frenkel, N. Reshetikhin, *Gaudin model, Bethe ansatz and correlation functions at the critical level*, Commun. Math. Phys. 166 (1), 27-62 (1995).
- [7] G. Felder, A. Varchenko, *Integral representation of solutions of the elliptic Knizhnik-Zamolodchikov-Bernard equations*, preprint hep-th/9502165.
- [8] E.V. Frenkel, *Affine algebras, Langlands duality and Bethe ansatz*, Proc. ICMP-94, 606-42, International Press (1995), (q-alg/9506003).
- [9] N. Nekrasov, *Holomorphic bundles and many-body systems*, preprint hep-th/95-03157.
- [10] E.K. Sklyanin, *Separation of variables in the Gaudin model*, J. Sov. Math. 47 (1989), 2473-88.

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